UDC 531.36

MULTIPLE SYNCHRONIZATION IN A SYSTEM OF WEAKLY-COUPLED OBJECTS WITH ONE DEGREE OF FREEDOM

PMM Vol. 36, №2, 1972, pp. 225-231
O. P. BARZUKOV
(Leningrad)
(Received July 8, 1971)

We investigate the periodic modes of motion in a system of objects with one degree of freedom, interacting through weak couplings, when in the absence of the couplings the objects perform periodic motions with frequencies which are the multiples of some value (the case of multiple synchronization [1]). We consider a system with a multidimensional rapidly-rotating phase, describing the interaction of nonlinear almost-conservative objects. We study the case, called "nonsimple" in [1], when the first-approximation periodicity conditions determine the connection only between the generating phases of the objects moving with like frequencies, while the connection between the phases of groups of objects having unequal frequencies are determined from the second-approximation periodicity conditions. We derive the necessary and sufficient stability conditions for the modes found. The multiple synchronization of mechanical vibrators in the "simple" case have been studied by the small-parameter method in [2]. A survey of other papers which study actual engineering objects by primarily asymptotic methods has been presented in [1].

1. Let us consider the problem of the interaction of nonlinear objects with weak couplings in the absence of an external force, described by the following system with a multidimensional rapidly-rotating phase:

$$\begin{gathered}\varphi_s^{\cdot} = \omega_s + \mu X_s^{(1)}(\varphi_1, \ldots, \varphi_n, \omega_1, \ldots, \omega_n, \mathbf{v}) + \\ + \mu^2 X_s^{(2)}(\varphi_1, \ldots, \varphi_n, \omega_1, \ldots, \omega_n, \mathbf{v}) + O(\mu^3) \\ \omega_s^{\cdot} = \mu Y_s^{(1)}(\varphi_1, \ldots, \varphi_n, \omega_1, \ldots, \omega_n, \mathbf{v}) + \\ + \mu^2 Y_s^{(2)}(\varphi_1, \ldots, \varphi_n, \omega_1, \ldots, \omega_n, \mathbf{v}) + O(\mu^3) \\ \mathbf{v}^{\cdot} = A\mathbf{v} + \mathbf{F}(\varphi_1, \ldots, \varphi_n, \omega_1, \ldots, \omega_n) + \mu \mathbf{F}^{(1)}(\varphi_1, \ldots, \varphi_n, \omega_1, \ldots, \omega_n, \mathbf{v}) + O(\mu^2)\end{gathered} \tag{1.1}$$

Here $\mu > 0$ is a small parameter, $\mathbf{v}$, $\mathbf{F}$, $\mathbf{F}^{(1)}$ are N-dimensional vectors, A is an $N \times N$ square matrix with constant coefficients; $X_s^{(1)}$, $X_s^{(2)}$, $Y_s^{(1)}$, $Y_s^{(2)}$, $\mathbf{F}$, $\mathbf{F}^{(1)}$ are assumed to be analytic in some region of the space of their arguments and to be 2π-periodic in each of the variables $\varphi_1, \ldots, \varphi_n$. The problem of the synchronization of quasi-conservative objects interacting through weak couplings reduces to equations of form (1.1) after a transition to "phase-frequency" variables [3, 4].

For $\mu = 0$ system (1.1) admits of a solution of the form

$$\varphi_s = \nu_s t + \alpha_s, \qquad \omega_s = \nu_s$$
$$\mathbf{v} = \mathbf{v}(\nu_1 t + \alpha_1, \ldots, \nu_n t + \alpha_n, \nu_1, \ldots, \nu_n) \tag{1.2}$$

where ν_s and α_s are constant frequencies and initial phases. For a complete system (1.1) we restrict ourselves to seeking the periodic solution, turning into solution (1.2) for $\mu = 0$ when all the ν_s are multiples of some frequency ν, i.e., $\nu_s = n_s\nu$. Suppose that there exist r groups of objects in each of which the n_s are the same; we denote the number of objects in the kth group by l_k. Without loss of generality the numbers n_k can be taken as not having common divisors.

We seek the periodic solution of system (1.1), of period $T = 2\pi / \nu$ in the form of a series in μ, taking (1.2) as the generating solution with due regard to the assumptions made. The frequency ν depends on μ and can be represented as the series

$$\nu = \nu^{(0)} + \mu\nu^{(1)} + \mu^2\nu^{(2)} + \ldots \tag{1.3}$$

whose coefficients are to be determined as we solve the problem. Instead of the variable t we introduce the variable t_1 by the formula

$$t_1 = (1 + \mu h^{(1)} + \mu^2 h^{(2)} + \ldots)\, t \qquad (h^{(i)} = \nu^{(i)} / \nu^{(0)}) \tag{1.4}$$

Then the problem reduces to seeking the $2\pi/\nu^{(0)}$-periodic solution of the new system

$$\varphi_{ki} = \omega_{ki} + \mu\,[X_{ki}^{(1)} - h^{(1)}\omega_{ki}] + \mu^2\,[X_{ki}^{(2)} - h^{(1)}X_{ki}^{(1)} - (h^{(2)} - h^{(1)^2})\,\omega_{ki}] + O(\mu^3)$$
$$\omega_{ki}' = \mu Y_{ki}^{(1)} + \mu^2\,[Y_{ki}^{(2)} - h^{(1)}Y_{ki}^{(1)}] + O(\mu^3)$$
$$\mathbf{v}' = A\mathbf{v} + \mathbf{F} + \mu\,[\mathbf{F}^{(1)} - h^{(1)}A\mathbf{v} - h^{(1)}\mathbf{F}] + O(\mu^2) \tag{1.5}$$

System (1.5) has been written in such a manner that the objects having like frequencies in the generating approximation are combined into one group with the index number k $(k = 1,\ldots, r)$, while i $(i = 1,\ldots, l_k)$ denotes the index number of the object within the group.

When solving system (1.5) by the small-parameter method, all the approximations for the functions φ_{ki} and ω_{ki} are determined, obviously, to within additive constants which can be considered as the corresponding higher-order components of the generating phases and frequencies. Therefore, right away we can seek the phases and frequencies in the form of series in μ with constant coefficients (see [5], Chap. III). Thus, finally, we seek the solution of system (1.5) in the form

$$\varphi_{ki} = n_k\tau + \alpha_{ki}^{(0)} + \mu\,(\varphi_{ki}^{(1)} + \alpha_{ki}^{(1)}) + \ldots$$
$$\omega_{ki} = \nu_{ki}^{(0)} + \mu\,(\omega_{ki}^{(1)} + \nu_{ki}^{(1)}) + \ldots, \qquad \mathbf{v} = \mathbf{v}^{(0)} + \mu\mathbf{v}^{(1)} + \ldots$$

where $\mathbf{v}^{(0)}$ represents the solution of the equation

$$\mathbf{v}^{(0)'} = A\mathbf{v}^{(0)} + \mathbf{F}\,(n_1\tau + \alpha_{11}^{(0)}, \ldots, n_r\tau + \alpha_{rl_r}^{(0)}) \tag{1.6}$$

which is 2π-periodic in the dimensionless time $\tau = \nu^{(0)}t_1$. The first-approximation periodicity conditions

$$P_{ki}^{(1)}\,(\alpha_{11}^{(0)}, \ldots, \alpha_{rl_r}^{(0)}, \nu^{(0)}, \nu_{11}^{(0)}, \ldots, \nu_{rl_r}^{(0)}) \equiv \frac{1}{2\pi}\int_0^{2\pi} (Y_{ki}^{(1)})\, d\tau = 0 \tag{1.7}$$

together with the conditions

$$\nu_{ki}^{(0)} = n_k\nu^{(0)} \qquad (i = 1, \ldots, l_k;\ k = 1, \ldots, r) \tag{1.8}$$

yield a system of equations for determining the generating phases $\alpha_{ki}^{(0)}$ and the initial approximation $\nu^{(0)}$ for the frequency. Here and subsequently, parantheses around functions and derivatives signify that they are computed at the generating solution.

If system (1.7) together with conditions (1.8) admits of a simple solution relative to the $l_1 + \ldots + l_r - 1$ phases (since the initial system is autonomous, one of the phases is arbitrary and can be taken equal to zero) and to the frequency $\nu^{(0)}$, then the multiple synchronization problem is practically not different from the simple synchronization problem [1]. Therefore, in what follows we examine the particular case, important for solving applied problems (especially the problem of the multiple synchronization of mechanical vibrators), when the functions $P_{ki}^{(1)}$ do not depend on those $\alpha_{pj}^{(0)}$ and $\nu_{pj}^{(0)}$ for which $p \neq k$

$$P_{ki}^{(1)} = P_{ki}^{(1)}(\alpha_{k1}^{(0)}, \ldots, \alpha_{kl_k}^{(0)}, \nu^{(0)}, \nu_{k1}^{(0)}, \ldots, \nu_{kl_k}^{(0)}) \tag{1.9}$$

In other words, we consider the case when from the first-approximation periodicity conditions there is established only the synchronization conditions for objects moving with like frequencies (the conditions for simple synchronization). Thus, system (1.7) breaks up into r independent systems each of which permits the determination of $l_k - 1$ generating phases to within additive constants $\alpha_k^{(0)}$, arbitrary in this approximation. The values of the frequency $\nu^{(0)}$, determined here from each system, are taken to be equal in correspondence with the assumptions made.

To determine the constants $\alpha_k^{(0)}$ we consider the following approximation to the unknown functions. From the second-approximation periodicity condition we obtain

$$P_{ki}^{(2)} \equiv \frac{1}{2\pi}\int_0^{2\pi}\Big\{\sum_{p=1}^{r}\sum_{j=1}^{l_p}\Big[\Big(\frac{\partial Y_{ki}^{(1)}}{\partial \varphi_{pj}}\Big)(\varphi_{pj}^{(1)} + \alpha_{pj}^{(1)}) + \Big(\frac{\partial Y_{ki}^{(1)}}{\partial \omega_{ki}}\Big)(\omega_{pj}^{(1)} + \nu_{pj}^{(1)})\Big] + \Big(\frac{\partial Y_{ki}^{(1)}}{\partial \nu}\Big)\nu^{(1)} - h^{(1)}(Y_{ki}^{(1)}) + (Y_{ki}^{(2)})\Big\}d\tau = 0 \tag{1.10}$$

where $P_{ki}^{(2)}$ are functions of all $\alpha_{pj}^{(1)}$ and $\nu_{p,}^{(1)}$ as well as of $\alpha_p^{(0)}$ not determined in the first approximation. The functions $P_{ki}^{(2)}$ can be represented as the sum of two terms, one of which does not depend on $\alpha_{pj}^{(1)}$, while the other is a linear function of $\alpha_{kj}^{(1)}$

$$P_{ki}^{(2)} = P_{*ki}^{(2)} + P_{**ki}^{(2)}$$

$$P_{*ki}^{(2)} = P_{*ki}^{(2)}(\alpha_1^{(0)}, \ldots, \alpha_r^{(0)}, \nu^{(1)}, \nu_{11}^{(1)}, \ldots, \nu_{rl_r}^{(1)})$$

$$P_{**ki}^{(2)} = \sum_{j=1}^{l_k}(\partial P_{ki}^{(1)} / \partial \alpha_{kj}^{(0)})\,\alpha_{kj}^{(1)}$$

Consequently, for finding the constants $\alpha_{kj}^{(1)}$ we have the r linear systems

$$\sum_{j=1}^{l_k}(\partial P_{ki}^{(1)} / \partial \alpha_{kj}^{(0)})\,\alpha_{kj}^{(1)} = -P_{*ki}^{(2)} \tag{1.11}$$

whose determinants equal zero. Therefore, for the systems to be solvable relative to $\alpha_{kj}^{(1)}$ it is necessary that the equalities

$$Q_k^{(2)}(\alpha_1^{(0)}, \ldots, \alpha_r^{(0)}, \nu^{(1)}, \nu_{11}^{(1)}, \ldots, \nu_{rl_r}^{(1)}) \equiv \sum_{i=1}^{l_k} P_{*ki}^{(2)} a_{ki}^{*} = 0 \tag{1.12}$$

be satisfied. In (1.12) the a_{ki}^* denote the solutions of the systems

$$\sum_{i=1}^{l_k} (\partial P_{ki}^{(1)} / \partial \alpha_{kj}^{(0)}) a_{ki}^* = 0 \qquad \begin{pmatrix} j = 1, \ldots, l_k \\ k = 1, \ldots, r \end{pmatrix} \tag{1.13}$$

System (1.12), supplemented by the conditions $\nu_{ki}^{(1)} = n_k \nu^{(1)}$, serves to determine the first approximation to the frequency and to the constants $\alpha_k^{(0)}$ one of which is arbitrary by virtue of the autonomous property of the system.

2. Going over to the investigation of the stability of the solutions found, we set up the variational equations of system (1.5). These equations represent a linear system with $(2\pi / \nu^{(0)})$ - periodic coefficients whose $2\ (l_1 + \ldots + l_r)$ characteristic indices vanish for $\mu = 0$. The stability of the solutions found is determined by the signs of the real parts of these so-called "critical" indices [4]. Therefore, we restrict ourselves only to looking for these. We represent the perturbations $\delta\varphi_{ki}$, $\delta\omega_{ki}$, $\delta\nu$ as

$$\delta\varphi_{ki} = e^{\lambda t_1}\vartheta_{ki}, \ \delta\omega_{ki} = e^{\lambda t_1}\psi_{ki}, \ \delta\nu = e^{\lambda t_1}w \tag{2.1}$$

where $\vartheta_{ki}(t_1)$, $\psi_{ki}(t_1)$, $w\ (t_1)$ are $(2\pi / \nu^{(0)})$ - periodic functions, while the critical indices $\lambda = \lambda(\mu)$ can be expanded into a series in powers of $\mu^{1/2}$

$$\lambda = \lambda_1 \mu^{1/2} + \lambda_2 \mu + \ldots \tag{2.2}$$

The successive approximations of the characteristic indices are sought during the construction of the $(2\pi / \nu^{(0)})$ - periodic solutions of the variational system as series in powers of $\mu^{1/2}$ with $(2\pi / \nu^{(0)})$ - periodic coefficients. As a result of the corresponding calculations, on which we do not dwell here, we arrive at the following results.

The first approximations to the characteristic indices are the roots of the equations

$$|\Lambda_{kij}| = 0, \ (\Lambda_{kij} = \partial P_{ki}^{(1)} / \partial \alpha_{kj}^{(0)} - \lambda_1^2 \delta_{ij}) \tag{2.3}$$

Each of these equations has a double zero root; the remaining root of each equation is assumed to be simple and nonzero. For the nonzero λ_1 the corresponding second approximations to the characteristic indices are [4]

$$\lambda_2 = \frac{1}{2} \sum_{i,\, j=1}^{l_k} \left[\frac{\partial P_{ki}^{(1)}}{\partial \nu_{kj}^{(0)}} + \frac{\partial R_{ki}^{(1)}}{\partial \alpha_{ki}^{(0)}} + \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{\partial Y_{ki}^{(1)}}{\partial \nu} \right) \zeta_{kj}\, d\tau \right] a_{ki}^* a_{kj}$$

$$\left(R_{kj}^{(1)} = \frac{1}{2\pi} \int_0^{2\pi} (X_{ki}^{(1)})\, d\tau \right) \tag{2.4}$$

where a_{ki} and a_{ki}^* are the solutions of the conjugate systems

$$\sum_{j=1}^{l_k} \Lambda_{kij} a_{kj} = 0, \qquad \sum_{i=1}^{l_k} \Lambda_{kij} a_{ki}^* = 0 \tag{2.5}$$

moreover, we assume that the vectors a_k and a_k^* are chosen so that

$$\sum_{i=1}^{l_k} a_{ki} a_{ki}^* = 1 \tag{2.6}$$

For those same characteristic indices for which the first approximations $\lambda_1 = 0$, the quantities λ_2 are determined from the equation

$$\left| \frac{\partial Q_k^{(2)}}{\partial \alpha_p^{(0)}} + \lambda_2 \sum_{i=1}^{l_k} \sum_{j=1}^{l_p} \left[\frac{\partial R_{ki}^{(1)}}{\partial \alpha_{pj}^{(0)}} + \frac{\partial P_{ki}^{(1)}}{\partial \nu_{pj}^{(0)}} + \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{\partial Y_{ki}^{(1)}}{\partial \nu} \right) \zeta_{pj} d\tau \right] a_{ki}^* - \right.$$
$$\left. - \lambda_2^2 \delta_{pk} \right| = 0 \qquad (k,\ p = 1, \ldots, r) \tag{2.7}$$

The functions ζ_{ki} in (2.4) and (2.7) are expressed in terms of $\nu^{(0)}$ in accordance with the relation [4]

$$\zeta_{ki}' = \nu^{(0)} \frac{\partial^2 \nu^{(0)}}{\partial \nu^{(0)} \partial \alpha_{ki}^{(0)}} \tag{2.8}$$

The multiple synchronization mode being investigated is stable if

$$\lambda_1^2 < 0, \qquad \operatorname{Re} \lambda_2 < 0$$

for all roots of Eqs. (2.3) and (2.7). Here one root of Eq. (2.7) equals zero, as must be for an autonomous system.

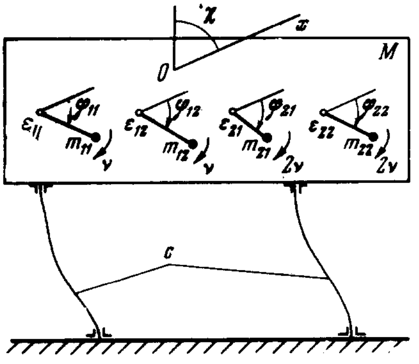

Fig. 1.

3. Let us consider the problem of the double synchronization of two pairs of unbalanced vibrators fixed on a rigid body which can accomplish translational oscillations (Fig. 1). The equations of motion of such a system have the form (see [1, 6] regarding the introduction of the small parameter)

$$\varphi_{ki}^{\cdot} = \omega_{ki}$$
$$\omega_{ki} = \mu \frac{1}{I_{ki}} [L_{ki}(\omega_{ki}) + m_{ki}\varepsilon_{ki} x^{\cdot\cdot} \sin \varphi_{ki} + m_{ki}\varepsilon_{ki} g \sin (\varphi_{ki} - \chi)] \qquad (k,\ i = 1, 2)$$
$$M x^{\cdot\cdot} + cx = \sum_{k,\ i=1}^{2} m_{ki}\varepsilon_{ki} (\omega_{ki}^{\cdot} \sin \varphi_{ki} + \omega_{ki}^2 \cos \varphi_{ki}) \tag{3.1}$$

where m_{ki}, ε_{ki}, I_{ki}, L_{ki} are the mass, the eccentricity, the moment of inertia, and the shaft torque of the ith unbalance in the kth group; M, c are the mass of the system and the rigidity of the elastic coupling, χ is the angle between the direction of the rigid body's motion and the vertical. The vibrators in each pair are considered alike, i.e.

$$m_{11} = m_{12} = m_1, \quad \varepsilon_{11} = \varepsilon_{12} = \varepsilon_1, \quad L_{11} = L_{12} = L_1, \quad I_{11} = I_{12} = I_1$$
$$m_{21} = m_{22} = m_2, \quad \varepsilon_{21} = \varepsilon_{22} = \varepsilon_2, \quad L_{21} = L_{22} = L_2, \quad I_{21} = I_{22} = I_2$$

We seek the existence conditions and the stability of the mode when the vibrators of the first pair rotate with an angular speed half that of the vibrators of the second pair. The first-approximation periodicity conditions for $2\nu_{11}^{(0)} = 2\nu_{12}^{(0)} = \nu_{21}^{(0)} = \nu_{22}^{(0)} = 2\nu^{(0)}$ lead to the equations

$$L_1(\nu^{(0)}) = 0, \qquad L_2(2\nu^{(0)}) = 0$$
$$\sin (\alpha_{12}^{(0)} - a_{11}^{(0)}) = 0, \qquad \sin (\alpha_{22}^{(0)} - \alpha_{21}^{(0)}) = 0 \tag{3.2}$$

For the existence of the mode being sought it is necessary that the equations for determining the frequencies admit of similar solutions. We further restrict ourselves to studying the case when the vibrators in each pair move coherently

$$\alpha_{11}^{(0)} = a_{12}^{(0)} = \alpha_1^{(0)}, \qquad \alpha_{21}^{(0)} = \alpha_{22}^{(0)} = \alpha_2^{(0)} \tag{3.3}$$

Then

$$Q_1^{(2)} = \frac{16 m_2 \varepsilon_2 m_1^2 \varepsilon_1^2 g \nu^{(0)^2}}{M I_1^2 (p^2 - 4\nu^{(0)^2})} \cos (2\alpha_1^{(0)} - \alpha_2^{(0)} - \chi) +$$

$$+ \frac{1}{2I_1} \left(\frac{dL_1}{d\omega_{11}}\right) \nu_{11}^{(1)} + \frac{1}{2I_1} \left(\frac{dL_1}{d\omega_{12}}\right) \nu_{12}^{(1)} = 0$$

$$Q_2^{(2)} = -\frac{8 m_2 \varepsilon_2 m_1^2 \varepsilon_1^2 g \nu^{(0)^2}}{M I_1 I_2 (p^2 - 4\nu^{(0)^2})} \cos (2\alpha_1^{(0)} - \alpha_2^{(0)} - \chi) + \frac{1}{2I_2} \left(\frac{dL_2}{d\omega_{21}}\right) \nu_{21}^{(1)} + \frac{1}{2I_2} \left(\frac{dL_2}{d\omega_{22}}\right) \nu_{22}^{(1)} = 0$$

$$2\nu_{11}^{(1)} = 2\nu_{12}^{(1)} = \nu_{21}^{(1)} = \nu_{22}^{(1)} = 2\nu^{(1)} \qquad (3.4)$$

Usually $(dL_k / d\omega_{ki}) < 0$ and, therefore, $\nu^{(1)} = 0$, and the equation connecting the constants $\alpha_1^{(0)}$ and $\alpha_2^{(0)}$ is obtained in the form

$$\cos (2\alpha_1^{(0)} - \alpha_2^{(0)} - \chi) = 0$$

whence $2\alpha_1^{(0)} - \alpha_2^{(0)} = \chi + \pi / 2$ or $2\alpha_1^{(0)} - \alpha_2^{(0)} = \chi + 3\pi / 2$.

From the stability conditions of the first group ($\lambda_1^2 < 0$) it follows that the phasing (3.4) is stable if $p^2 > 4\nu^{(0)^2}$ Relations (2.4) lead to the requirement [6] that $(dL_k/d\omega_{ki}) < 0$. Finally, the condition that the roots of Eq. (2 7) be negative shows that for $p^2 > 4\nu^{(0)^2}$ the mode is stable when $2\alpha_1^{(0)} - \alpha_2^{(0)} = \chi + 3\pi / 2$. We take the initial phases of the rotation of the vibrators of the first pair, $\alpha_{11}^{(0)}$ and $\alpha_{12}^{(0)}$ equal to zero. Then the law of oscillations $x(t)$ of the rigid body is

$$x = \frac{2 m_1 \varepsilon_1 \nu^2}{M (p^2 - \nu^2)} \cos \nu t + \frac{8 m_2 \varepsilon_2 \nu^2}{M (p^2 - 4\nu^2)} \cos \left(2\nu t - \chi - \frac{3\pi}{2}\right) + O(\mu) =$$

$$= \frac{2 m_1 \varepsilon_1 \nu^2}{M (p^2 - \nu^2)} \cos \nu t - \frac{8 m_2 \varepsilon_2 \nu^2}{M (p^2 - 4\nu^2)} \sin (2\nu t - \chi) + O(\mu) \qquad (3.5)$$

where $\nu = \nu^{(0)} + O(\mu^2)$ The other solutions of Eq. (3.2) are investigated analogously.

The author thanks I. I. Blekhman for discussing the results and for valuable advice.

BIBLIOGRAPHY

1. Blekhman, I. I., Synchronization of Dynamical Systems. Moscow, "Nauka", 1971.
2. Zaretskii, L. B., Multiple synchronization of centrifugal vibration exciter. Izv. Akad. Nauk SSSR, MTT, №4, 1971.
3. Nagaev, R. F., General problem of synchronization in an almost conservative system. PMM Vol. 29, №5, 1965.
4. Nagaev, R. F. and Khodzhaev, K. Sh., Synchronous motions in a system of objects with supporting constraints. PMM Vol. 31, №4, 1967.
5. Malkin, I. G., Certain Problems in the Theory of Nonlinear Oscillations. Moscow Gostekhizdat, 1966.
6. Khodzhaev, K. Sh., Synchronization of mechanical vibrators coupled with a linear oscillatory system. Inzh. Zh. MTT, №4, 1967.

Translated by N. H. C.